Scaling laws and dispersion equations for Lévy particles in one-dimensional fractal porous media

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Over some range of scales natural porous media often display a fractal Eulerian velocity or conductivity field. If one assumes the fractal conductivity gives rise to fractal drift velocities, then particle paths may be studied in the framework of stochastic differential equations (SODEs). On the microscale, trajectories are modeled as solutions to a SODE with Markovian, stationary, ergodic drift subject to a fluctuating Lévy force. The Lévy force allows for self-motile particles such as flagellated microbes. On the mesoscale the trajectories are modeled as solutions to a SODE with Lévy (fractal) drift and diffusion arising from the microscale asymptotics. On the macroscale the process is driven by the asymptotics of the mesoscale drift without diffusion. Asymptotic scaling laws and dispersion equations are presented.

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I. INTRODUCTION

Nature abounds with natural porous media [1,2] such as tissues, cells, whole plants and animals, oil reservoirs, soils, and even the earth's upper mantle. Processes occurring in such systems act from the atomic scale [3], where electrons propagate through crystalline materials to the intermediate scale, where blood profuses in the lung [4] to the global scale, where magma migrates in the lithosphere [5]. Many of these systems display a fractal character functionally over some range of scales [6]. Typical examples are profusion in the lung and groundwater flow [7].

In this contribution we examine the behavior of motile particles in porous systems that have fractal functionality between lower and upper fractal cutoffs (see Fig. 1). Below the lower fractal cutoff (microscale or pore scale) we consider the flow to be classical in the sense that the path line of a particle will be considered to be the solution to a stochastic ordinary differential equation (SODE) with stationary, ergodic, Markov drift velocity, and to account for particles that may be self-motile, such as microbes [8], we allow for diffusion driven by an α-stable Lévy process [9–11]. If $\alpha=2$ the Lévy process becomes Brownian and the particles may be inert.

Between the fractal cutoffs (mesoscale or Darcy scale) the porous formation is assumed to have fractal functionality dictated by the Eulerian velocity field. If the system is Darcian with a linear constitutive relation between the convective flux and gradient of potential (the coefficient of proportionality is called the Darcy conductivity), then the conductivity is assumed to have a fractal spatial structure [7] which in turn induces the spatially fractal Eulerian velocity. If the mesoscale flow is considered incompressible and steady, then the drift velocity of a particle will be the Eulerian velocity at the spatial point x when the Lagrangian particle $X(t)$ satisfies $X(t;x_0)=x$. That is, $V(t)=v(X(t;x_0))$, $X(0)=x_0$, where $v(x)$ is the Eulerian velocity field and $V(t)$ is the Lagrangian drift velocity. Under these conditions the mesoscale drift velocity will in general be statistically nonstationary but may possess stationary increments. The diffusive structure at the mesoscale is obtained from the asymptotics of the microscale process.

On the macroscale we assume there is no additional drift; that is, the physics are controlled solely by the asymptotic behavior of the mesoscale process. We assume that above the upper fractal cutoff the system is periodic [1].

The outline of the manuscript is as follows. In Sec. II we review the classical properties of a Lévy process and use these to develop the characteristic function for the integral of such a process. Using this, we show that a displacement process with Lévy Lagrangian velocity has integer fractal dimension and that the diffusion (Fokker-Planck) equation for the transition density is spatially nonlocal but temporally lo-

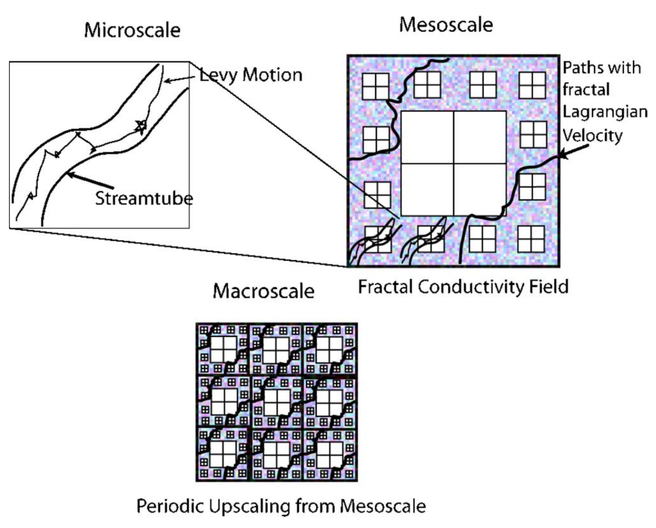

FIG. 1. (Color online) Gedanken experiment: micro-, meso-, and macroscale domains; shaded, high conductivity, and white, low conductivity.

cal with time dependent dispersion coefficient.

Next we return to the multiscale problem for Secs. III–VII. Section III defines the microscale stochastic differential equation that represents the temporal evolution of a motile particle in a classical convective flow field within a pore. A classical central limit theorem for a Brownian process is coupled to a scaling law for Lévy processes and used to upscale the particle trajectory from the pore to Darcy scale (micro- to mesoscale). The resultant process is Lévy. In Sec. IV the mesoscale Lévy diffusive process is combined with a Lévy convective flux that represents the Lagrangian convective velocity through the mesoscale fractal porous body. Scaling laws and the associated scale change are derived. In Sec. V, by assuming a periodic macroscale we homogenize the process at that scale. Section VI uses the characteristic function for the upscaled process to derive a fractional diffusion equation for the transition density with a time-dependent diffusion coefficient in the macroscale. This equation is subsequently written in conservative form.

II. LÉVY PROCESS

Consider the following stochastic ordinary differential equation in one dimension:

$$X(t) = X(0) + \int_0^t V(r)dr + \rho L(t), \quad t \geq 0, \tag{2.1}$$

where the stochastic processes $\{V(r)\}$ and $\{L(r)\}$ are α-stable Lévy processes [9,10] and ρ is a constant.

Let $\widetilde{V}(t)=V(t)-V(0)$ and $\widetilde{X}(t)=X(t)-X(0)$. Then, for each t, $\widetilde{V}(t)$ has an α-stable distribution [9] $[V(t) \sim S_{\alpha_v}(t^{1/\alpha_v}\sigma_v, \beta_v, t\mu_v)]$, whose characteristic function ϕ_v is

$$\phi_v(t,\theta) = \exp\left\{ - t\sigma_v^{\alpha_v}|\theta|^{\alpha_v}\left[1 - i\beta_v \operatorname{sgn}(\theta)\tan\left(\frac{\alpha_v \pi}{2}\right)\right] + i\theta t\mu_v \right\}, \tag{2.2}$$

where $0<\alpha_v \leq 2$, $\alpha_v \neq 1$, $\beta_v \in [-1,1]$, $\sigma_v > 0$, and $\operatorname{sgn}(\theta) = 1$ if $\theta>0$, 0 if $\theta=0$, and -1 otherwise. For $\alpha_v=1$, we take $\beta=0$:

$$\phi_v(t,\theta) = \exp(-t\sigma_v|\theta| + i\theta\mu_v t). \tag{2.3}$$

Notationally we write the characteristic function for $L(t)$ the same as for $V(t)$ with the subscript v replaced by ℓ. Let

$$Y(t) = \int_0^t V(r)dr.$$

Then $Y(t)$ can be decomposed as

$$Y(t) = \int_0^t [V(0) + \widetilde{V}(r)]dr = tV(0) + \widetilde{Y}(t),$$

where $\widetilde{Y}(t)=\int_0^t \widetilde{V}(r)dr$.

For each t, the characteristic function ϕ_y of $\widetilde{Y}(t)$ is

$$\phi_y(t,\theta) = \begin{cases} \exp\left\{ -\dfrac{t^{1+\alpha_v}\sigma_v^{\alpha_v}}{1+\alpha_v}|\theta|^{\alpha_v}\left[1 - i\beta_v \operatorname{sgn}(\theta)\tan\left(\dfrac{\pi\alpha_v}{2}\right)\right] + i\theta\dfrac{\mu_v t^2}{2}\right\}, & \alpha_v \neq 1, \\ \exp\left(-\dfrac{t^2\sigma_v}{2}|\theta| + i\theta\dfrac{\mu_v t^2}{2}\right), & \alpha_v = 1. \end{cases} \tag{2.4}$$

To see this, argue as follows. For simplicity the index v for each parameter of the characteristic function is dropped. We consider $\alpha \in (0,1) \cup (1,2]$. The same argument can be applied to the case $\alpha=1$ with $\beta=0$. We calculate the characteristic function of $\{\widetilde{Y}(t)\}$ and investigate the distribution of each increment. It is well known that Lévy processes have a countable number of discontinuities, so the integral on time may be considered in the Riemann sense:

$$\widetilde{Y}(t) = \lim_{n\to\infty}\sum_{j=1}^{n}\frac{t}{n}\widetilde{V}(r_j),$$

where $0=r_0<r_1<\cdots<r_n=t$ and $r_j - r_{j-1} = t/n$ for $j=1,2,\ldots,n$. Since each α-stable random variable is represented by its characteristic function, we compute the characteristic function ϕ_y of $\widetilde{Y}(t)$:

$$\phi_y(t,\theta) = E\{\exp[i\theta\widetilde{Y}(t)]\} = \lim_{n\to\infty} E\left[\exp\left(i\theta\sum_{j=1}^{n}\frac{t}{n}\widetilde{V}(r_j)\right)\right].$$

Let

$$A_n = E\left[\exp\left(i\theta\sum_{j=1}^{n}\frac{t}{n}\widetilde{V}(r_j)\right)\right];$$

then

$$\begin{aligned} A_n &= E\left[\prod_{j=1}^{n}\exp\left(i\theta\frac{t}{n}\widetilde{V}(r_1)\right)\exp\left(i\theta\frac{t}{n}[\widetilde{V}(r_2)-\widetilde{V}(r_1)]\right)\cdots \times\exp\left(i\theta\frac{t}{n}[\widetilde{V}(r_j)-\widetilde{V}(r_{j-1})]\right)\right] \\ &= E\left[\exp\left(i\theta n\frac{t}{n}\widetilde{V}(r_1)\right)\exp\left(i\theta(n-1)\frac{t}{n}[\widetilde{V}(r_2)-\widetilde{V}(r_1)]\right)\cdots \times\exp\left(i\theta\frac{t}{n}[\widetilde{V}(r_n)-\widetilde{V}(r_{n-1})]\right)\right]. \end{aligned}$$

Since $\widetilde{V}(r_1)$, $[\widetilde{V}(r_2)-\widetilde{V}(r_1)],\ldots,[\widetilde{V}(r_n)-\widetilde{V}(r_{n-1})]$ are independent, and so are $\exp[i\theta n(t/n)\widetilde{V}(r_1)],\exp\{i\theta(n-1)(t/n)\times[\widetilde{V}(r_2)-\widetilde{V}(r_1)]\},\ldots,\exp\{i\theta(t/n)[\widetilde{V}(r_n)-\widetilde{V}(r_{n-1})]\}$. Therefore,

$$A_n = E\{\exp[i\theta t\widetilde{V}(r_1)]\}E\left[\exp\left(i\theta t\frac{n-1}{n}[\widetilde{V}(r_2)-\widetilde{V}(r_1)]\right)\right]\cdots \times E\left[\exp\left(i\theta\frac{t}{n}[\widetilde{V}(r_n)-\widetilde{V}(r_{n-1})]\right)\right].$$

Since $\{\widetilde{V}(t)\}$ has stationary increments, we obtain

$$A_n = E\{\exp[i\theta t\widetilde{V}(r_1)]\} \times E\left[\exp\left(i\theta t\frac{n-1}{n}\widetilde{V}(r_1)\right)\right]\cdots E\left[\exp\left(i\theta\frac{t}{n}\widetilde{V}(r_1)\right)\right] = \prod_{j=0}^{n-1}\phi_v\left(\frac{t}{n},\frac{n-j}{n}\theta t\right),$$

where $r_1=t/n$. Then,

$$\phi_y(t,\theta) = \lim_{n\to\infty}\prod_{j=0}^{n-1}\phi_v\left(\frac{t}{n},\frac{n-j}{n}\theta t\right).$$

We take the natural logarithm:

$$\ln\phi_y(t,\theta) = \lim_{n\to\infty}\sum_{j=0}^{n-1}\ln\phi_v\left(\frac{t}{n},\frac{n-j}{n}\theta t\right)$$

and note that

$$\ln\phi_v\left(\frac{t}{n},\frac{n-j}{n}\theta t\right) = -t^{1+\alpha}\sigma^\alpha|\theta|^\alpha\left[1-i\beta\,\mathrm{sgn}(\theta)\times\tan\left(\frac{\pi\alpha}{2}\right)\right] \frac{(n-j)^\alpha}{n^{1+\alpha}} + i\theta\mu t^2\frac{n-j}{n^2}.$$

Hence,

$$\ln\phi_y(t,\theta) = -t^{1+\alpha}\sigma^\alpha|\alpha|^\alpha\left[1-i\beta\,\mathrm{sgn}(\theta)\tan\left(\frac{\pi\alpha}{2}\right)\right] \times\lim_{n\to\infty}\sum_{j=0}^{n-1}\frac{1}{n}\left(1-\frac{j}{n}\right)^\alpha + i\theta\mu t^2\lim_{n\to\infty}\sum_{j=0}^{n-1}\frac{1}{n}\left(1-\frac{j}{n}\right).$$

The infinite sums can be exactly evaluated:

$$\lim_{n\to\infty}\sum_{j=0}^{n-1}\frac{1}{n}\left(1-\frac{j}{n}\right)^\alpha = \int_0^1(1-x)^\alpha dx = \frac{1}{1+\alpha}$$

giving the desired result.

We investigate properties of the process $\{\widetilde{Y}(t)\}$ further. For each $t_0>0$, the natural logarithm of the characteristic function of $\widetilde{Y}(t)-\widetilde{Y}(t_0)$, $t>t_0$, is

$$\ln\phi_y^{t_0}(t,\theta) = -t_0(t-t_0)^{\alpha_v}\sigma_v^{\alpha_v}|\theta|^{\alpha_v}\left[1-i\beta_v\mathrm{sgn}(\theta)\tan\left(\frac{\pi\alpha_v}{2}\right)\right] + i\theta\mu_v t_0(t-t_0) - \frac{(t-t_0)^{1+\alpha_v}\sigma_v^{\alpha_v}}{1+\alpha_v}|\theta|^{\alpha_v} \times\left[1-i\beta_v\mathrm{sgn}(\theta)\tan\left(\frac{\pi\alpha_v}{2}\right)\right] + i\theta\mu_v\frac{(t-t_0)^2}{2}. \quad (2.5)$$

The above shows that $\widetilde{Y}(t)$ does not have stationary increments.

We next examine the fractal dimension of $\widetilde{Y}(t)$. There are a variety of fractal dimensions in use such as box dimension, divider dimension, Hausdorff dimension, etc. We refer to [6,12–14] for the details of the dimensions. It is well known that the divider D_D and box D_B dimensions of an α-stable Lévy process are α and $\max\{1,2-1/\alpha\}$ [13], respectively. When these dimensions are computed, stationarity of the increments of Lévy processes is used. Although $\widetilde{Y}(t)$ does not have stationary increments, we can still calculate the box dimension of the process.

Consider for a fixed $T>0$, $\{\widetilde{Y}(t):0\leqslant t\leqslant T\}$. Let $\Delta t=T/m$, $T_k=k\Delta t$, $k=0,1,2,\ldots,m$, and for each k define the random variable R_k by

$$R_k = \max\{|\widetilde{Y}(s)-\widetilde{Y}(r)|:T_{k-1}\leqslant r<s\leqslant T_k\}.$$

Equation (2.5) gives

$$\widetilde{Y}(s)-\widetilde{Y}(r)\overset{d}{=}\left(r+\frac{s-r}{1+\alpha}\right)^{1/\alpha}(s-r)Z,$$

where $\overset{d}{=}$ means equality in distribution, and Z is a random variable with α-stable distribution: $Z\sim S_{\alpha_v}(\sigma,\beta,0)$. Thus,

$$R_k\overset{d}{=}|Z|\max\{g(r,s):T_{k-1}\leqslant r<s\leqslant T_k\},$$

where

$$g(r,s) = \left(r+\frac{s-r}{1+\alpha}\right)^{1/\alpha}(s-r).$$

Let $\Delta t=bt$ with a scaling factor b and a fixed time $t>0$. Cover the graph $\{(t,\widetilde{Y}(t)):0\leqslant t\leqslant T\}$ of $\widetilde{Y}(t)$ by the rectangular boxes with the dimension $(ba)\times(bt)$, where a is a fixed positive real number. We compute the expected value of the number of boxes $N^{\Delta t}$. Since in each subinterval $[T_{k-1},T_k]$, the number of boxes is $R_k/(ba)$,

$$E[N^{\Delta t}] = \sum_{k=1}^{m}\frac{E[R_k]}{(ba)}. \quad (2.6)$$

To compute the numerator of the right hand side of (2.6), note that

$$E[R_k] = E[|Z|]\max g(r,s)$$

and $(\partial g/\partial r)(r,s)<0$ and $(\partial g/\partial s)(r,s)>0$ for any positive real numbers r and s $(r<s)$. Thus,

$$E[R_k] = E[|Z|]g(T_{k-1},T_k) = E[|Z|](bt)^{1+1/\alpha}\left(k - \frac{\alpha}{1+\alpha}\right)^{1/\alpha} \tag{2.7}$$

and hence

$$E[N^{bt}] = E[|Z|]\frac{(bt)^{1+1/\alpha}}{(ba)}\sum_{k=1}^{m}\left(k - \frac{\alpha}{1+\alpha}\right)^{1/\alpha}$$
$$= t^{1+1/\alpha}a^{-1}b^{1/\alpha}E[|Z|]\sum_{k=1}^{m}\left(k - \frac{\alpha}{1+\alpha}\right)^{1/\alpha}.$$

Let $C(t,\alpha)=t^{1+1/\alpha}a^{-1}E[|Z|]$. Observe that

$$\sum_{k=1}^{m}(k-1)^{1/\alpha} \leqslant \sum_{k=1}^{m}\left(k - \frac{\alpha}{1+\alpha}\right)^{1/\alpha} \leqslant \sum_{k=1}^{m}k^{1/\alpha}. \tag{2.8}$$

Since

$$\sum_{k=1}^{m}k^{1/\alpha} = m^{1+1/\alpha}\sum_{k=1}^{m}\frac{1}{m}\left(\frac{k}{m}\right)^{1/\alpha} \approx m^{1+1/\alpha}\int_0^1 x^{1/\alpha}dx$$
$$= \frac{\alpha}{1+\alpha}m^{1+1/\alpha}$$

for sufficiently large m, the sums of the left and right hand sides of Eq. (2.8) are the lower and upper sums of the integral. Thus,

$$\sum_{k=1}^{m}\left(k - \frac{\alpha}{1+\alpha}\right)^{1/\alpha} \approx \frac{\alpha}{1+\alpha}m^{1+1/\alpha} = \frac{\alpha}{1+\alpha}\left(\frac{T}{bt}\right)^{1+1/\alpha} \tag{2.9}$$

and hence

$$E[N^{bt}] \approx \frac{\alpha}{1+\alpha}C(t,\alpha)\left(\frac{T}{t}\right)^{1+1/\alpha}b^{-1} = O(b^{-1}).$$

Since $E[N^{bt}]=O(b^{-D_B})$, we obtain that the box dimension of $\widetilde{Y}(t)$ is 1 if $\mu_v=0$.

Benson [15] and Meerschaert *et al.* [16] have shown that the density functions of Lévy motions are the solutions of fractional Fokker-Planck equations. Since $\widetilde{Y}(t)$ is a time integral of a Lévy motion, a natural question arises: What is the Fokker-Planck equation for the process $\widetilde{Y}(t)$?

The forward fractional derivative $d^\alpha/(dx^\alpha)$ is defined by

$$\frac{\partial^\alpha f}{\partial x^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_0^\infty \frac{f(x-y)}{y^{\alpha+1-n}}dy, \tag{2.10}$$

where $n-1\leqslant\alpha<n$ with an integer n [16].

Let $f_{\widetilde{Y}}(t,x)$ be the density functions of $\widetilde{Y}(t)-\mu_v t^2/2$ for each t. The characteristic function of $\widetilde{Y}(t)-\mu_v t^2/2$ is

$$\hat{f}_{\widetilde{Y}}(t,\theta) = \exp\left\{-\frac{t^{1+\alpha_v}\sigma_v^{\alpha_v}}{1+\alpha_v}|\theta|^{\alpha_v}\left[1 - i\beta_v \mathrm{sgn}(\theta)\tan\left(\frac{\pi\alpha_v}{2}\right)\right]\right\}, \tag{2.11}$$

where $\hat{f}_{\widetilde{Y}}$ is the Fourier transform of $f_{\widetilde{Y}}$. By differentiating $\hat{f}_{\widetilde{Y}}$ with respect to t, we have

$$\frac{\partial \hat{f}_{\widetilde{Y}}}{\partial t} = -\frac{(t\sigma_v)^{\alpha_v}}{\cos(\pi\alpha_v/2)}|\theta|^{\alpha_v} \times\left[\cos\left(\frac{\pi\alpha_v}{2}\right) - i\beta_v\mathrm{sgn}(\theta)\sin\left(\frac{\pi\alpha_v}{2}\right)\right]\hat{f}_{\widetilde{Y}}. \tag{2.12}$$

If $\alpha_v=2$, then we obtain the classical Fokker-Planck equation by taking the Fourier transform:

$$\frac{\partial f_{\widetilde{Y}}}{\partial t} = D(t)\frac{\partial^2 f_{\widetilde{Y}}}{\partial x^2},$$

where $D(t)=(t\sigma_v)^2$. For the general case, by employing the identity

$$(-i\theta)^{\alpha_v} = |\theta|^{\alpha_v}\left[\cos\left(\frac{\pi\alpha_v}{2}\right) - i\,\mathrm{sgn}(\theta)\sin\left(\frac{\pi\alpha_v}{2}\right)\right],$$

setting $\beta_v=1$, and taking Fourier transforms, we find

$$\frac{\partial f_{\widetilde{Y}}}{\partial t} = D(t)\frac{\partial^{\alpha_v} f_{\widetilde{Y}}}{\partial x^{\alpha_v}}, \tag{2.13}$$

where

$$D(t) = \frac{-(t\sigma_v)^{\alpha_v}}{\cos(\pi\alpha_v/2)}.$$

Remark. If $1<\alpha_v\leqslant 2$, then $\cos(\pi\alpha_v/2)<0$ and so $D(t)>0$.

Remark. Dividing Eq. (2.13) by t^{α_v} gives another form of Fokker-Planck equation:

$$t^{-\alpha_v}\frac{\partial f_{\widetilde{Y}}}{\partial t} = D\frac{\partial^{\alpha_v} f_{\widetilde{Y}}}{\partial x^{\alpha_v}}, \tag{2.14}$$

where

$$D = \frac{-\sigma_v^{\alpha_v}}{\cos(\pi\alpha_v/2)}.$$

III. UPSCALING FROM MICROSCALE TO MESOSCALE

On the microscale we assume the SODE is of the form

$$X^{(0)}(t) = X^{(0)}(0) + \int_0^t V^{(0)}[X^{(0)}(r)]dr + \rho^{(0)}L^{(0)}(t), \quad t\geqslant 0, \tag{3.1}$$

where $V^{(0)}[X^{(0)}(r)]$ is assumed stationary, ergodic, and Markovian with mean $V^{(0)}$, and $L^{(0)}(t)$ is an α_l-stable Lévy process $[L^{(0)}(t)\sim S_{\alpha_l}(t^{1/\alpha_l}\sigma_1,\beta_l,t\mu_l)]$ which accounts for the self-motility (in the case of a microbe) of the particle [8]. Here, the superscript 0 means microscale. We use 1 for mesoscale and 2 for macroscale. If $\alpha_l=2$, then $L^{(0)}(t)$ reduces to $B^{(0)}(t)$, a Brownian motion, and the particle need not be motile. Let $\tilde{X}^{(0)}(t)=X^{(0)}(t)-X^{(0)}(0)$.

By using the classical central limit theorem [17] it has been shown that as $n\to\infty$,

$$n^{-1/2}\int_0^{nt}\{V^{(0)}[X^{(0)}(r)]-\bar{V}^{(0)}\}dr \xrightarrow{d} B^{(1)}(t), \qquad (3.2)$$

a Brownian motion, where $\xrightarrow{d}$ means convergence in distribution.

We next show $[\tilde{X}^{(0)}(nt)-nt(\bar{V}^{(0)}+\mu_l)]/n^{1/\alpha_l}$ converges to an α_l-stable Lévy process in distribution as $n\to\infty$.

Clearly for $\rho^{(0)}$ a constant, as $n\to\infty$,

$$\frac{\rho^{(0)}L^{(0)}(nt)-nt\mu_l}{n^{1/\alpha_L}} \xrightarrow{d} \rho^{(1)}L^{(1)}(t), \qquad (3.3)$$

where $L^{(1)}(t)\sim S_{\alpha_l}(t^{1/\alpha_l}\sigma_l,\beta_l,0)$.

In order to use Eqs. (3.2) and (3.3), we look at

$$\frac{\tilde{X}^{(0)}(nt)-nt(\bar{V}^{(0)}+\mu_l)}{n^{1/\alpha_l}} = \frac{1}{n^{1/\alpha_l-1/2}}\frac{1}{n^{1/2}}\int_0^{nt}\{V^{(0)}[X^{(0)}(r)] - \bar{V}^{(0)}\}dr + \frac{\rho^{(0)}L^{(0)}(nt)-nt\mu_l}{n^{1/\alpha_l}}. \qquad (3.4)$$

By Eq. (3.2), the first term in the right hand side converges to zero in distribution if $\alpha_l\neq 2$. Thus, Eq. (3.3) gives the result for $\alpha_l\neq 2$. If $\alpha_l=2$, then the two terms of the right hand side converge to Brownian motions, respectively. Since the sum of two Brownian processes is Brownian, we obtain the result.

Remark. For $t>0$, $\tilde{X}^{(1)}(t)\approx t(\bar{V}^{(0)}+\mu_l)+\rho^{(1)}L^{(1)}(t)$.

IV. SCALING LAWS AND SCALE CHANGE FOR MESOSCALE CONVECTION

Let $V^{(1)}(t)$ be α_v-stable Lévy with $1<\alpha_v\leqslant 2$. We assume that $V^{(1)}(t)$ has constant mean and $V^{(1)}(0)$ has initial distribution π_v.

$$E(V^{(1)}(t))=E(V^{(1)}(0))=\bar{V}^{(1)} \quad \text{for any } t\geqslant 0 \qquad (4.1)$$

if $\alpha_v>1$. Since $E(\tilde{V}^{(1)}(t))=0$ for each time t, $\mu_v^{(1)}=0$. Thus, $\tilde{V}^{(1)}(t)\sim S_{\alpha_v}(t^{1/\alpha_v}\sigma_v,\beta_v,0)$.

Let $t=\lambda t'$. Define a new random variable

$$W_\lambda^{(1)}(t') \equiv \frac{Y^{(1)}(\lambda t')-\lambda t'\bar{V}^{(1)}}{\lambda^{1+1/\alpha_v}} = \lambda^{-1/\alpha_v}t'(V^{(1)}(0)-\bar{V}^{(1)})+\lambda^{-1-1/\alpha_v}\tilde{Y}^{(1)}(\lambda t'). \qquad (4.2)$$

Since $\lambda^{-1/\alpha_v}t'(V^{(1)}(0)-\bar{V}^{(1)})$ converges to zero in probability as $\lambda\to\infty$, it also converges in distribution. For the second term of the right hand side of Eq. (4.2), we compute the natural logarithm of its characteristic function:

$$\begin{aligned}\ln E\{\exp[i\theta\lambda^{-1-1/\alpha_v}\tilde{Y}^{(1)}(\lambda t')]\} &= -\frac{\lambda^{1+\alpha_v}}{1+\alpha_v}\sigma_v^{\alpha_v}|\lambda^{-1-1/\alpha_v}\theta|^{\alpha_v}t'^{\,1+\alpha_v}\times\left[1-i\beta_v\mathrm{sgn}(\theta)\tan\left(\frac{\pi\alpha_v}{2}\right)\right]\\ &= -\frac{t'^{\,1+\alpha_v}}{1+\alpha_v}\sigma_v^{\alpha_v}|\theta|^{\alpha_v}\left[1-i\beta_v\mathrm{sgn}(\theta)\tan\left(\frac{\pi\alpha_v}{2}\right)\right]. \end{aligned} \qquad (4.3)$$

Note that Eq. (4.3) is independent of λ. Let

$$\tilde{Y}^{(2)}(t)=\lim_{\lambda\to\infty}\tilde{Y}^{(1)}(\lambda t). \qquad (4.4)$$

We have shown that $\tilde{Y}^{(2)}(t)\approx\tilde{Y}^{(1)}(t)$, whose characteristic function is determined by Eq. (4.3). Thus, for $t>0$ we have

$$Y^{(2)}(t)\approx t\tilde{V}^{(1)}+\tilde{Y}^{(2)}(t). \qquad (4.5)$$

V. TOTAL UPSCALING

Since $\rho^{(1)}L^{(1)}(0)=0$, for fixed t, $\rho^{(1)}L^{(1)}(t)$ has an α_ℓ-stable distribution: $\rho^{(1)}L^{(1)}(t)\sim S_{\alpha_\ell}(t^{1/\alpha_\ell}\sigma_\ell,\beta_\ell,t\mu_\ell^{(1)})$.

We assume periodicity of $V^{(1)}(t)=V^{(1)}(t,\xi^{(1)}(0))$ in the initial data:

$$V^{(1)}(t,\xi^{(1)}(0)+\nu)=V^{(1)}(t,\xi^{(1)}(0))$$

for any integer ν, where $\xi^{(1)}(\cdot)=\xi^{(1)}(\cdot,\omega)$ is a particle path in the porous medium.

Let

$$[V^{(1)}]=E\left[\frac{1}{|A|}\int_A\frac{1}{|B|}\int_B V^{(1)}(t,\xi^{(1)}(0))dt\,d\xi^{(1)}(0)\right],$$

where $A=A(\omega)=\{\xi^{(1)}(0,\omega):\xi^{(1)}(0,\omega)\in[0,1]\}$ and $B=B(\omega)=\{t\geqslant 0:\xi^{(1)}(t,\omega)\in[0,1]\}$.

Our goal is to arrive at a central limit theorem for $\tilde{X}^{(1)}(t)$. To do so, let

$$Z_\lambda^{(1)}(t') \equiv \frac{\tilde{X}^{(1)}(\lambda t')-\lambda t'([V^{(1)}]+\mu_\ell^{(1)})}{\lambda^{1+1/\alpha_v}} = \frac{t'(\bar{V}^{(1)}-[V^{(1)}]-\mu_\ell^{(1)})}{\lambda^{1/\alpha_v}}+\frac{Y^{(1)}(\lambda t')-\lambda t'\bar{V}^{(1)}}{\lambda^{1+1/\alpha_v}} + \frac{\rho^{(1)}L^{(1)}(\lambda t')}{\lambda^{1+1/\alpha_v}}. \qquad (5.1)$$

As $\lambda\to\infty$, the first and second terms of the right hand side of Eq. (5.1) converge to zero and $\tilde{Y}^{(2)}(t')$ in distribution, respectively. Consider the last term on the right hand side of Eq. (5.1). Let

$$Q_\lambda^{(1)}(t') \equiv \frac{\rho^{(1)}L^{(1)}(\lambda t')}{\lambda^{1+1/\alpha_v}};$$

then the natural logarithm of the characteristic function of $Q_\lambda^{(1)}(t')$ is

$$\begin{aligned}\ln E\{\exp[i\theta\lambda^{-1-1/\alpha_v}\rho^{(1)}L^{(1)}(\lambda t')]\} &= -\lambda^{1-\alpha_\ell-\alpha_\ell/\alpha_v}t'\sigma_\ell^{\alpha_\ell}|\theta|^{\alpha_\ell}\left[1-i\beta_\ell\text{sgn}(\theta)\tan\left(\frac{\pi\alpha_\ell}{2}\right)\right]\\ &\quad + i\lambda^{-1/\alpha_v}t'\theta\mu_\ell^{(1)},\end{aligned} \tag{5.2}$$

where $\alpha_\ell \neq 1$. When $\alpha_\ell = 1$ ($\beta=0$) it is easily proved that the characteristic function converges to zero for each θ as $\lambda \to \infty$. To obtain pointwise convergence of the characteristic function, we have to make the exponent of λ negative. Note that

$$1-\alpha_\ell - \frac{\alpha_\ell}{\alpha_v} < 0 \Leftrightarrow \alpha_v(1-\alpha_\ell) < \alpha_\ell. \tag{5.3}$$

We summarize the result. If $2/3 < \alpha_\ell \leq 2$, then for each α_v, $Z_\lambda^{(1)}(t')$ converges to $\tilde{Y}^{(2)}(t')$ in distribution. If $1/2 < \alpha_\ell \leq 2/3$, then we have convergence under the constraint

$$1 < \alpha_v < \frac{\alpha_\ell}{1-\alpha_\ell}. \tag{5.4}$$

Define $\tilde{X}^{(2)}(t)$ by

$$\tilde{X}^{(2)}(t) = \lim_{\lambda\to\infty}\tilde{X}^{(1)}(\lambda t). \tag{5.5}$$

Remark. For $t > 0$, we can approximate the stochastic process $\tilde{X}^{(2)}(t)$ as

$$\tilde{X}^{(2)}(t) \approx t([V^{(1)}] + \mu_\ell^{(1)}) + \tilde{Y}^{(2)}(t). \tag{5.6}$$

Thus, $\tilde{X}^{(2)}(t)$ has the same fractal dimension as $\tilde{Y}^{(2)}(t)$.

VI. ADVECTION-DISPERSION EQUATIONS

The next goal is to derive the Fokker-Planck (advection-dispersion) equation that the transition density for $\tilde{X}^{(2)}(t)$ satisfies.

For each t, $\tilde{X}^{(2)}(t)$ has an α-stable distribution with a density function $f(x,t)$. Therefore, we consider the natural logarithm of the Fourier transform of $f(x,t)$:

$$\begin{aligned}\ln\hat{f}(\theta,t) &= itv^{(2)}\theta - \frac{t^{1+\alpha_v}}{1+\alpha_v}\sigma_v^{\alpha_v}|\theta|^{\alpha_v}\left[1-i\beta_v\text{sgn}(\theta)\tan\left(\frac{\pi\alpha_v}{2}\right)\right].\end{aligned} \tag{6.1}$$

Since $\alpha_v \neq 1$, we can use the approach of [16]:

$$\begin{aligned}\frac{\partial\hat{f}}{\partial t} &= -\frac{(t\sigma_v)^{\alpha_v}}{\cos(\pi\alpha_v/2)}|\theta|^{\alpha_v}\left[\cos\left(\frac{\pi\alpha_v}{2}\right)\right.\\ &\quad \left. - i\beta_v\text{sgn}(\theta)\sin\left(\frac{\pi\alpha_v}{2}\right)\right]\hat{f} + v^{(2)}(i\theta\hat{f}).\end{aligned} \tag{6.2}$$

By Fourier inversion, we find that if

$$1 < \alpha_v < \frac{\alpha_\ell}{1-\alpha_\ell} \quad \text{and} \quad \frac{1}{2} < \alpha_\ell \leq \frac{2}{3}, \tag{6.3a}$$

$$1 < \alpha_v < 2 \quad \text{and} \quad \frac{2}{3} < \alpha_\ell \leq 2 \tag{6.3b}$$

for $\beta_v = 1$ and $\alpha_v = 2$ for $\beta_v = 0$, then $f(x,t)$ is the solution of the advection-dispersion equation

$$\frac{\partial f}{\partial t} = -v^{(2)}\frac{\partial f}{\partial x} + D^{(2)}(t)\frac{\partial^{\alpha_v}f}{\partial x^{\alpha_v}}, \tag{6.4}$$

where $v^{(2)} = [V^{(1)}] + \mu_\ell^{(1)}$, $D^{(2)}(t) = -(t\sigma_v)^{\alpha_v}/\cos(\pi\alpha_v/2)$, and $d^\alpha/(dx^\alpha)$ is defined by

$$\frac{\partial^\alpha f}{\partial x^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_0^\infty \frac{f(x-y)}{y^{\alpha+1-n}}dy, \tag{6.5}$$

where $n-1 \leq \alpha < n$ with an integer n [18].

Cushman and Moroni [19] have shown that the fractional advection-dispersion equation is a special case of their convolution-Fickian nonlocal advection-dispersion equation. According to their approach, we can derive the following conservative form from Eqs. (6.4) and (6.5):

$$D^{(2)}(t)\frac{\partial^{\alpha_v}f}{\partial x^{\alpha_v}} = \frac{d}{dx}\int_0^t\int_{-\infty}^\infty \tilde{D}^{(2)}(y,t,\tau)\frac{df(x-y,t-\tau)}{d(x-y)}dy\, d\tau, \tag{6.6}$$

where $1 < \alpha_v \leq 2$ and

$$\tilde{D}^{(2)}(y,t,\tau) = \frac{D^{(2)}(t-\tau)\delta(\tau)H(y)}{\Gamma(n-\alpha_v)y^{\alpha_v+1-n}}, \tag{6.7}$$

with the Dirac delta function $\delta(\tau)$ and the Heaviside function $H(y)$.

VII. DISCUSSION

Many natural porous media display a fractal functionality over some range of scales. One-dimensional porous media with fractal drift velocity on an intermediate scale have been studied. On the microscale, particle trajectories were the solution to a SODE with stationary, ergodic, Markov drift velocity and Lévy diffusion. The Lévy diffusion allows for the study of self-motile particles like flagellated microbes. The mesoscale fractal Eulerian velocity was assumed to give rise to a fractal Lagrangian drift velocity with stationary increments. This velocity was modeled as a Lévy process because given a fractal with independent increments, one can generate a Lévy motion with the same fractal dimension and statistical character. The mesoscale diffusion process was determined by the asymptotics of the microscale and the mesoscale flow was assumed steady and incompressible. The macroscale physics is controlled solely by the mesoscale asymptotics. The mesoscale flow field was mapped to a periodic lattice to obtain the macroscale field.

Scaling laws in terms of the stability parameter of the Lévy processes were obtained at both the meso- and macroscales. Fractional dispersive equations with positive dispersion coefficient were also obtained at these scales. In form, these latter equations are similar to the Fokker-Planck equations for Lévy processes except for the time dependence of the dispersion coefficient.

There is significant difference between this work and previous efforts; the drift velocity is assumed fractal as opposed to the drift trajectory (streamtube). This is consistent with experimental observations [20] for many natural systems.

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